$$Hv\left(s_1 :: s_{le}\right) \approx \lim_{t \to \infty} \sum_{N_1,,N_{le-1}} \left\lfloor \frac{e^{-h \sum_{p=1}^{le-1} \left(\ln \frac{s_p}{s_{le}} - \ln \frac{N_p}{t} \right)^2}}{hv\left(N_1 :: N_{le-1} : t\right)} \right\rfloor$$

A Mathematical Theory of Sensory Harmonics

by Gus Wiseman

The problem of aesthetics is one that has occupied many philosophical minds, from Plato to Kant to Poe. In fact, it has generally been considered a philosophical study. Neverthess, certain mathematicians, such a Pythagoras, Euler, and Birkhoff, have attempted to approach the subject from a mathematical point of view. Their success has been limited, however, due to the "fuzzy" nature of aesthetics. In fact it may be said with reasonable credibility that no major advancement has been made in the mathematical study of aesthetics since the time of Pythagoras, over 2000 years ago.

A throrough definition of aesthetics as this paper defines it, is necessary. Aesthetics is not limited to visual arts, nor is it limited to art at all. Aesthetics can be musical, poetic, literary, it can be found in nature or in the human body, aesthetics is that sensual input which pleases the mind.

This paper describes and discuesses the theory of sensory harmonics. We shall say that harmonics describes the relations between the numerical ratios of simple aesthetic objects, such as musical tones or multi-dimensinal bricks. Consonance, then, describes those harmonics which have high order value. Dissonance describes those with low value.

The theory of sensory harmonics is highly dependant upon the principle of harmonic distance. In standard terms, the distance between two values is equivelent to the distance between those same two values, manifested physically. For example, the distance between 2 and 5 is equivalent to the distance between a point 2 units from a source and a point 5 units from the same source in the same direction.

Mathematically, the distance between two values s and v, is:

$$d(s,v) = |s - v|$$

or

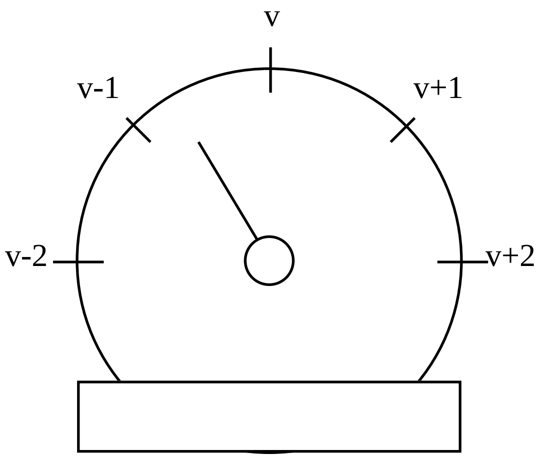
$$d(s,v) = \sqrt{(s-v)^2}$$

Therefore, the distance between a value s and zero is:

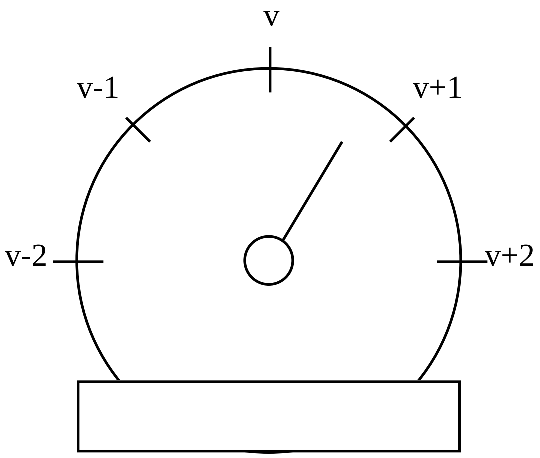
$$d(s,0) = \sqrt{s^2}$$

This definition of distance is applicable to many forms of visual analysis. For example, if we look at a meter, with a needle indicating a value, the distance between the needle and a certain value is represented by the above equations, where the needle's location is represented by s, and the value by v.

If we assume the meter to be symetrical across a vertical axis, then the needle's base position is in the center, whose value we will assign to v. We can then say that for any two values of s, where $d(s_1,v)=d(s_2,v)$, the meter is visually equivalent, as in the case below:



s=v-.7

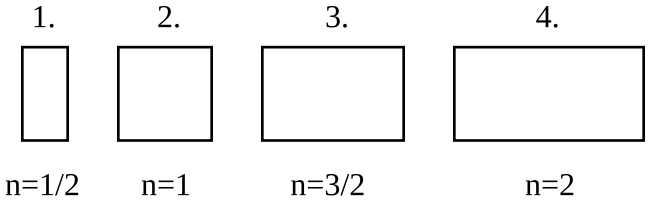


s=v+.7

$$d(s_1, v) = d(v - .7, v) = |v - .7 - v| = .7$$
$$d(s_2, v) = d(v + .7, v) = |v + .7 - v| = .7$$

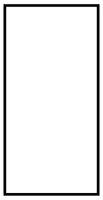
In this way, we can declare visual equivalence if the distances between the two values and a symetrical origin are equal.

This form is not applicable, however, to other forms of visual analysis, and to all forms of audible analysis. Take, for instance, a rectangle. If we exclude the factor of size, any rectangle may be represented by a ratio between two integers, or a rational number *n*.



Because we are excluding the factor of size, and therefore consider rectangles of similar ratios to be equal, regardless of size, we could also show the rectancle 1 at a different scale. When placed next to rectangle 4, it is clear that they are visually equivelent.

1.



n=1/2

4.

n=2

According to the formula given earlier, the distance between these two rectangles and the origin (*n=1*) are 1/2 and 1, respectively. Thus, even though the rectangles are visually equivelent, their distances are not equal.

Perhaps this point is better illustrated audibally. Two tones played at specific frequencies can be desribed by the ratio of the frequencies. In music an octave represents a ratio of 2:1. Thus if the frequency of A is 220, then the frequency of the next As will be 440, 880, 1760, etc. However, audibally, a high A played with a higher A, are equivalent to a low A played with a lower A, despite the fact the the distances between higher As are greater than those between lower As according to distance formula above.

Clearly a new formula for *harmonic distance (Hd)* is necessary. We know that the harmonic distance of 2 (when only one variable is given for a distance function, it is assumed that that it represents the distance between that variable and the origin, which is 1 for harmonic distance and usually 0 for standard distance) is equal to the harmonic distance of 1/2, or:

$$Hd(2) = Hd\left(\frac{1}{2}\right)$$

In broader terms:

$$Hd(s) = Hd\left(\frac{1}{s}\right)$$

$$Hd\left(\frac{s}{v}\right) = Hd\left(\frac{v}{s}\right)$$

$$Hd(s,v) = Hd(v,s)$$

$$Hd(s,v) = Hd(ks,kv)$$

$$Hd\left(\frac{s}{v}\right) = Hd\left(\frac{s}{v},1\right) = Hd\left(\frac{s(v)}{v},1(v)\right) = Hd(s,v)$$

The challenge is finding an equation for which these conditions are true. It is clear that Hd(0)=∞, so the equation is isymptotic. It is also clear that Hd(1)=0. With some further analysis, it becomes clear that the desired function is logarithmic:

Theorom 1A.

$$Hd(s,v) = |\ln s - \ln v| = \left|\ln\left(\frac{s}{v}\right)\right| = \sqrt{\ln^2\left(\frac{s}{v}\right)}$$

It is important to note that the base for the logarithm is not necessarily *e* for this and the other equations in this paper to work, it is used merely out of convenience.

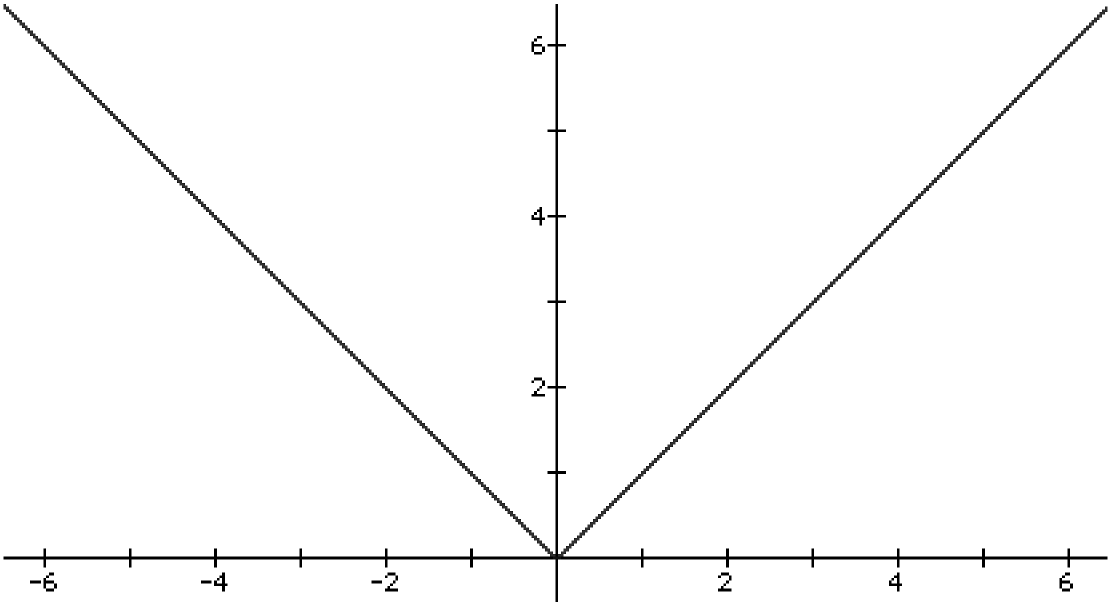
On furthar analysis, we can see why this function works:

$$\ln\frac{s}{v} = \ln sv^{-1} = \ln^{-1} s^{-1}v = -\ln s^{-1}v = -\ln\frac{v}{s}$$

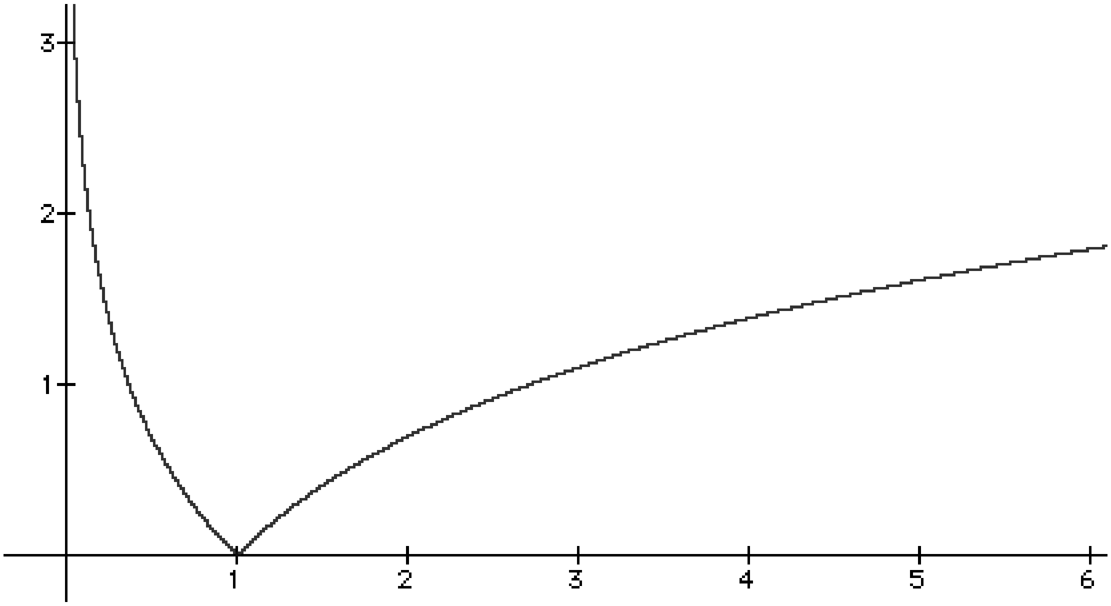
It is important to note the similarity between the final equation and the original distance equation:

$$d(s,v) = |s - v|$$

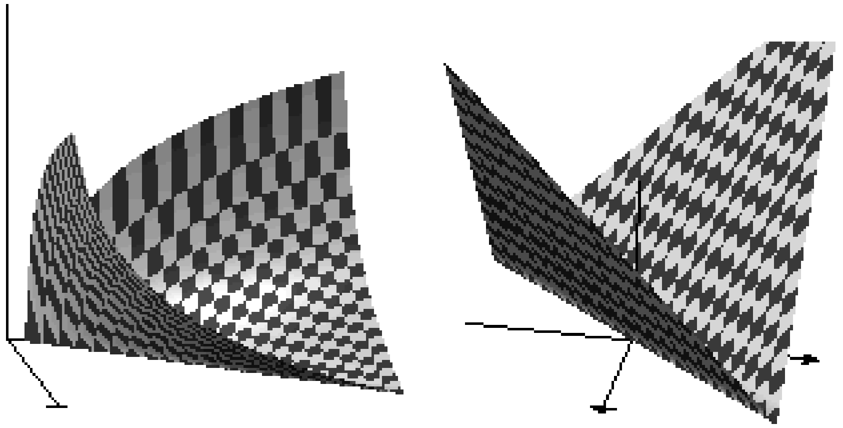
$$Hd(s,v) = |\ln s - \ln v|$$



$d(x) = |x|$



$Hd(x) = |\ln x|$



$$Hd(x,y) = |\ln x - \ln y|, d(x,y) = |x - y|$$

The purpose of obtaining this formula lies in the broader problem of aesthetic value. Many mathematicians have created partial solutions the problem of aesthetics, but none are as accurate as that which is described in this paper, to the knowledge of the author.

Pythagoras was among the first to seek a mathematical solution to aesthetics. His research focused on the harmonic relationship between musical tones, or consonace. He found that a suitable scale of tones could be created using ratios using only twos and threes as factors. In other words, all the tones of a scale can be described by:

$$2^a 3^b$$

For example:

$$\frac{9}{8} = 2^{-3}3^2, \frac{4}{3} = 2^2 3^{-1}, \ldots$$

Euler developed a theory which described the harmonic value of two tones played together as depending on the size of the prime factors of the ratio of the tones. If we assume an integer to be in prime factor form (where p_n represents the nth prime and k represents its number of appearances):

$$p_1{}^{k_1} \cdot p_2{}^{k_2} \cdot \ldots \cdot p_n{}^{k_n}$$

Then the harmonic value (Euler’s degree of sweetness) of an integer is:

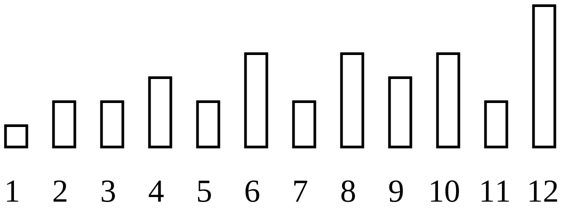
$$(p_1 - 1)k_1 + (p_2 - 1)k_2 + \ldots + (p_n - 1)k_n$$

By studying the formula it becomes clear that the result will be lower if only low primes are used, and higher if high primes (or many primes) exist in the integer.

To account for ratios as well as integers, Euler said:

$$V\left(\frac{a}{b}\right) = V(a) + V(b)$$

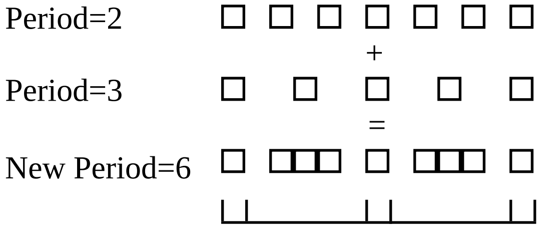
Ray Thomes pointed out a sequence that tends to correspond to harmonic values. This sequence is generated by counting the number of divisors had by an integer. For example, 12 has 6 divisors, 1, 2, 3, 4, 6, and 12.



Thomes shows that, if one studies this series (especially between 24 and 48), the higher points correspond to the ratios of modern scales.

Another common view is that consonance is based upon overtones. Overtones are increasingly inaudible tones which are heard when a tone is played on an instrument. They exist at twice the frequency of the tone, three times the frequency of the tone, four times, five times, etc. Thus, when two tones are played together, certain overtones happen to be the same. For example, if a 220hz tone is played with a 330hz tone, the third overtone of the first aligns with the second overtone of the second, thus producing a consonant tone. This is illustrated in the work of Norman Sohl.

My own research has led to a different theory. When two tones are played together, their sound waves are added together. Thus, if a 220hz frequency is added to a 330hz frequency, the length of the overall tone has increased. More simply, if two periodic functions are added, one with a period of 2 and the other with 3, the new period will be longer (6):



The new period length will be the least common multiple of the original lengths. It is important to note, however, that the new period length is relative to the other lengths. LCM(2,2) is two, even though the ratio of 2:2 is equivalent to 1:1 and LCM(1,1) is 1. To account for this the original ratio must be in minized terms. Any ratio can be minimized using its greatest common denominator:

$$m(s{:}v) = \frac{s}{\gcd(s,v)} : \frac{v}{\gcd(s,v)}$$

Thus the dissonance of two tones is approximated by:

Theorom 3.

$$hv(s,v) = lcm\left(\frac{s}{\gcd(s,v)}, \frac{v}{\gcd(s,v)} \right)$$

This theorom considers a ratio with greater elements to be “longer” and thus more dissonant. Ratios with smaller elements are “shorter” and more consonant. This theorom is an approximation because it is very difficult to obtain solid experimental data to prove or disprove it or the other theories given above. Additionally, it shares in a flaw that exists in every mathematical theory of consonance we have discussed.

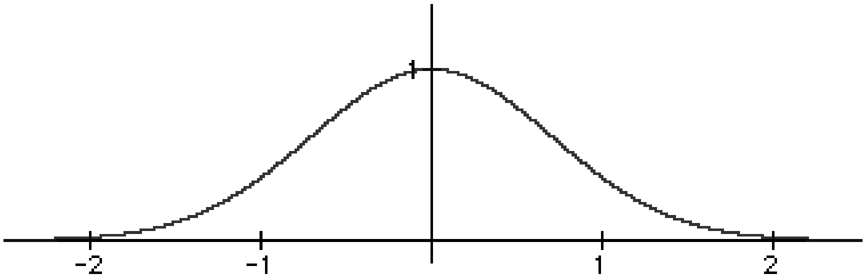
The ratio of 2:1 is relatively consonant. This is shown by every theory of consonance presented. However, the ratio of 2001:1000 is also consonant, almost as consonant as 2:1. However, every theory of consonance presented would consider it to be very dissonant. This is the flaw and major roadblock in the development of a complete mathematical thoery of consonance.

It can be said that 2001:1000 is a consonant ratio because it is near to 2:1, which is consonant. In other words, the *harmonic distance* between 2001:1000 and 2:1 is small. In fact, the ear is not sensetive enough to make much of a distinction between the two.

Clearly, it is necessary to derive a formula which describes the audible likeness of two tones. This formula will not be equal to the harmonic distance between the tones for several reasons. Note that D(1/2,1)=-0.6931 and D(1/4,1)=-1.386. However, 1/2:1 does not sound any more like 1:1 than 1/4:1 does. In fact, 1/2:1 sounds only a very little bit more like 1:1 than 1/4:1 does (in reality, neither 1/2:1 nor 1/4:1 sound at all like 1:1, but if our ears were less sensetive, they would.) This proves that the function we seek will flatten out at its ends, as in a bell curve. Additionally, it will not be isymptotic at its peak, because if it were, 1:1 would be infinitely more like to 1:1 than 1001:1000 is to 1:1. Obviosly, 1001:1000 is almost just a like to 1:1 as 1:1 is.

To determine the necessary equation, we will return to the meter problem and determine how much one value for the point of the meter is like another value indicated by the meter. Accoding to Wayne Hild, this equation can probably be described using a normal bell curve. We will say that *h* represents the sensetivity of the observer; 2000 is a typical value for music.

$$Sn(s,v) = e^{-h\cdot(d(s,v))^2}$$

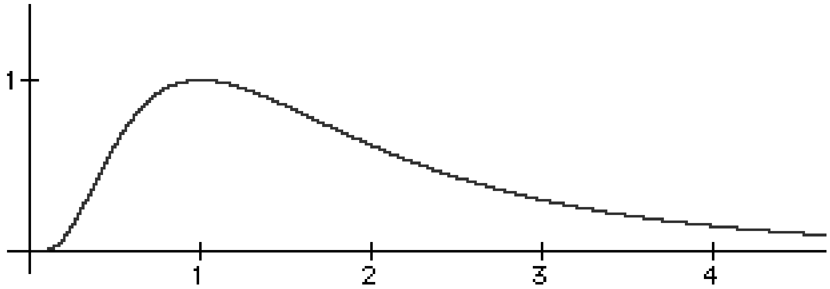


$$Sn(x) = e^{-hx^2}$$

Thus if *Sn*(*s*,*v*) represents standard likeness, *Hn*(*s*,*v*) will represent harmonic likeness:

Theorom 4A.

$$Hn(s,v) = e^{-h\cdot(Hd(s,v))^2}$$



$$Hn(x) = e^{-h\cdot\ln^2 x}$$

Now the only remaining task is to create a formula for *Hv*, or the true consonance of two tones. Historically, it has been said that *Hv*=*hv*, where *hv* is an approximation for the consonance of a ratio. *Hv* does not equal *hv* however, due to the 2001:1000 example discussed previously. In fact, *Hv* of two tones in relation to a certain ratio, or $Hvs(s,N_s)$ is:

$$Hvs(s, N_s) = Hn(s, N_s)\cdot hv(N_s)$$

In words, the consonance of two tones with respect to a certain ratio is equal to the harmonic nearness of the ratios times the consonance of the that ratio. In reality, our ears compare two tones heard together to every ratio. In other words, we account for every possible ratio of N_S. Thus:

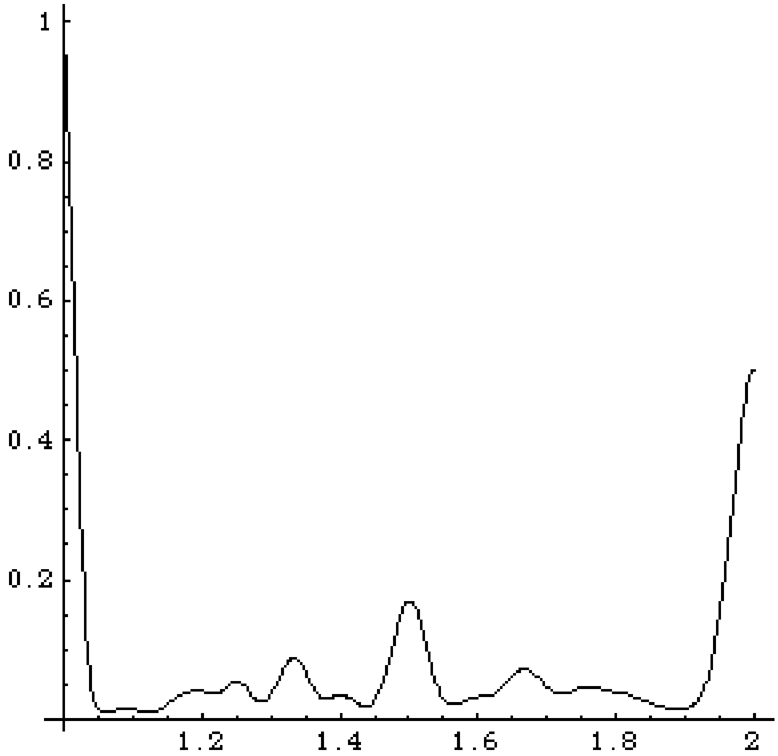
Theorom 5A.

$$Hv(s) = \lim_{t\to\infty} \sum_{N_s} \left\lfloor Hn\left(s, \frac{N_s}{t} \right) \cdot hv\left(\frac{N_s}{t} \right) \right\rfloor$$

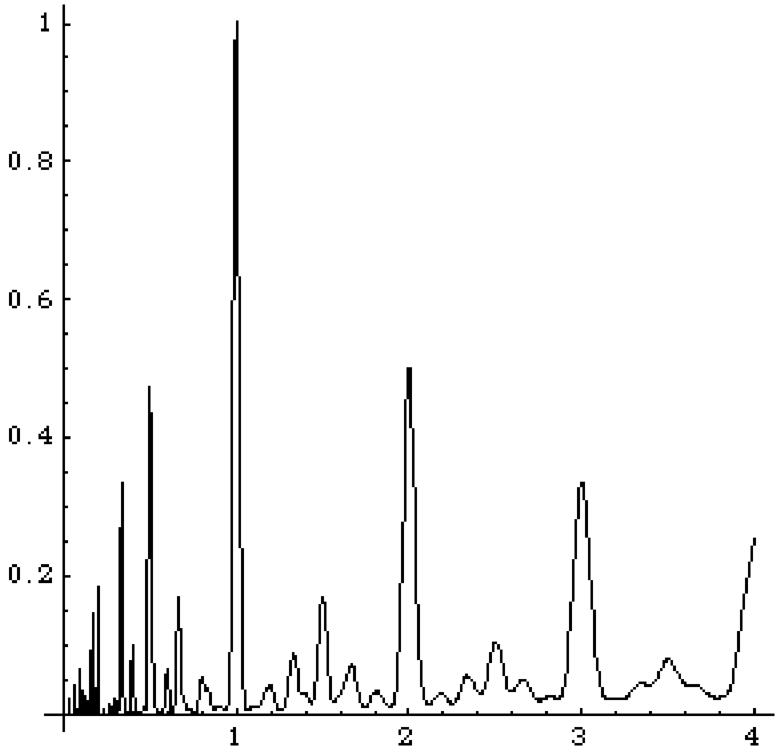
We can now expand this formula using Theoroms 3 and 4. We will also invert it to represent consonance rather than dissonance.

Theorom 6A.

$$Hv(s) = \lim_{t\to\infty} \sum_{N_s} \left\lfloor \frac{e^{-h \ln^2 \frac{s \cdot t}{N_s}}}{lcm\left(\frac{N_s}{\gcd(N_s, t)}, \frac{t}{\gcd(N_s, t)} \right)} \right\rfloor$$



$h = 2000, t = 60$



$$h = 2000, t = 30$$

These graphs represent the harmonic value of two tones played together whose ratio is represented by the *x*-axis (*s*). They correspond excellently to reality.

The next problem is that of consonance of more than two tones. We shall now develop a broader theory of harmonic value, one which can decide the consonance of more complex chords, such as triads.

We have established that the distance between two numbers is represented by:

$$d(s,v) = \sqrt{(s-v)^2}$$

We shall say that these numbers contain two components each, where a component is an element of a ratio:

$$d(s,v) = d(s{:}1, v{:}1)$$

If we assume a cartesian definition of distance (as in the standard distance formula in geometry) we can generalize this formula:

$$d\left(s_1{:}s_2, v_1{:}v_2\right) = \sqrt{\left(s_1 - v_1\right)^2 + \left(s_2 - v_2\right)^2}$$

If the number of components is *le* and we notate a multiple component ratio as:

$$s_1{:}s_2{:}\mathbf{L}{:}s_{le} = s_1{::}s_{le}$$

Then:

$$d\left(s_1{::}s_{le}, v_1{::}v_{le}\right) = \sqrt{\sum_{p=1}^{le}\left(s_p - v_p\right)^2}$$

Thus:

$$Hd\left(s_1{::}s_{le-1}{:}1, v_1{::}v_{le-1}{:}1\right) = \sqrt{\sum_{p=1}^{le-1}\left(\ln s_p - \ln v_p\right)^2}$$

To broaden the formula for non-grounded ratios, we can use the identity:

$$s_1{:}s_2 = \frac{s_1}{s_2}{:}1$$

$$Hd\left(s_1{::}s_{le}, v_1{::}v_{le}\right) = \sqrt{\sum_{p=1}^{le-1}\left(\ln\frac{s_p}{s_{le}} - \ln\frac{v_p}{v_{le}}\right)^2}$$

And:

Theorom 1B.

$$Hn\left(s_1{::}s_{le}, v_1{::}v_{le}\right) = e^{-h\left(Hd\left(s_1{::}s_{le}, v_1{::}v_{le}\right)\right)^2}$$

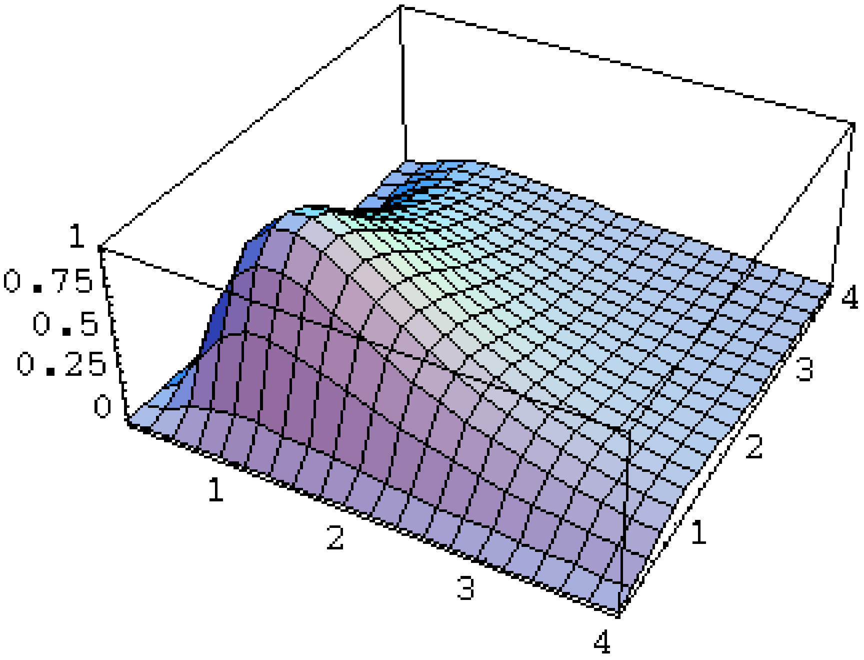
In the case of triads (*le*=3), we have:

$$Hn(s_1:s_2:s_3,1:1:1)=e^{-h\left(\ln^2\frac{s_1}{s_3}+\ln^2\frac{s_2}{s_3}\right)}$$

We can assume that, as before, the ratio is grounded, so that s_3 shall be 1:

Theorom 4B.

$$Hn(s_1:s_2:1,1:1:1)=e^{-h\left(\ln^2 s_2+\ln^2 s_2\right)}$$



$$Hn(x:y:1,1:1:1)=e^{-h\left(\ln^2 x+\ln^2 y\right)}$$

We can now revise our definition of harmonic value:

Theorom 5B.

$$Hv(s_1::s_{le})=\lim_{t\to\infty}\sum_{N_1,\mathbf{L},N_{le-1}}\left\lfloor\frac{Hn(s_1::s_{le},N_1::N_{le-1}:t)}{hv(N_1::N_{le-1}:t)}\right\rfloor$$

Our estimation of *hv* can also be revised:

$$hv(s_1::s_{le}) = lcm\left(\frac{s_1}{\gcd(s_1,,s_{le})},\mathbf{L},\frac{s_{le}}{\gcd(s_1,,s_{le})}\right)$$

Now Theorom 6 can be recreated:

Theorom 7.

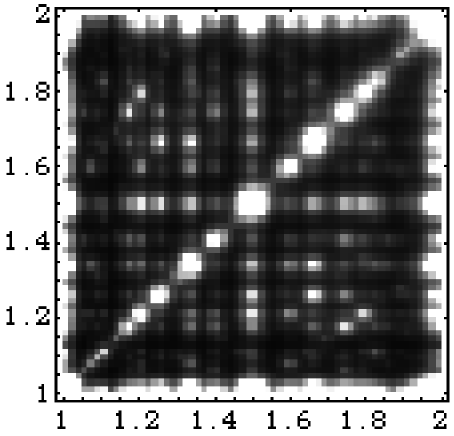
$$Hv(s_1::s_{le}) \approx \lim_{t\to\infty}\sum_{N_1,,N_{le-1}}\left\lfloor \frac{e^{-h\sum\limits_{p=1}^{le-1}\left(\ln\frac{s_p}{s_{le}}-\ln\frac{N_p}{t}\right)^2}}{hv(N_1::N_{le-1}:t)}\right\rfloor$$

Thus for triads:

$$Hv(s_1:s_2:s_3) \approx \lim_{t\to\infty}\sum_{N_1,N_2}\left\lfloor \frac{e^{-h\left(\left(\ln\frac{s_1}{s_3}-\ln\frac{N_1}{t}\right)^2+\left(\ln\frac{s_2}{s_3}-\ln\frac{N_2}{t}\right)^2\right)}}{hv(N_1:N_2:t)}\right\rfloor$$

With a grounded ratio:

$$Hv(s_1:s_2:1) \approx \lim_{t\to\infty}\sum_{N_1,N_2}\left\lfloor \frac{e^{-h\left(\left(\ln s_1-\ln\frac{N_1}{t}\right)^2+\left(\ln s_2-\ln\frac{N_2}{t}\right)^2\right)}}{hv(N_1:N_2:t)}\right\rfloor$$



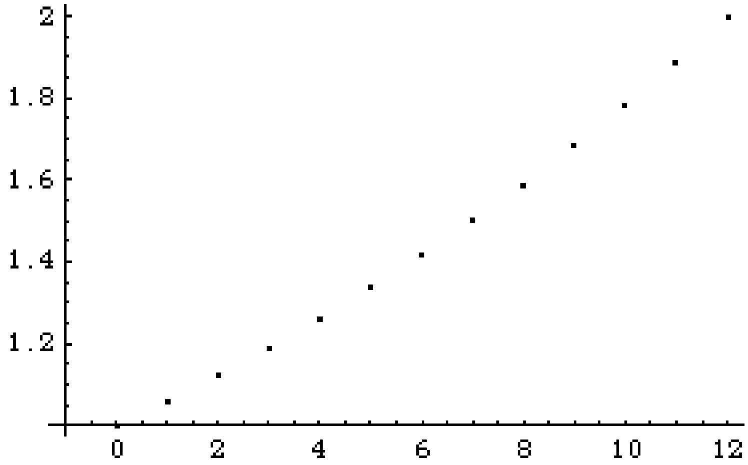
h=2000, *t*=60

It is difficult to test or apply the ideas we have discussed to music without first transposing them to modern scales and musical ideas. The modern western scale consists of twelve tones which can be said to increase in ratio from 1 to 2. These tones are repeated to create a scale of several octaves. The distance between two tones in this scale varies, but the ratio (and harmonic distance) between two neighboring tones always remains the same. Thus, it is called the equitempered scale.

If $R(n)$ is the ratio of note n to the fundamental (note 0), then one can generate the ratios in this scale as follows:

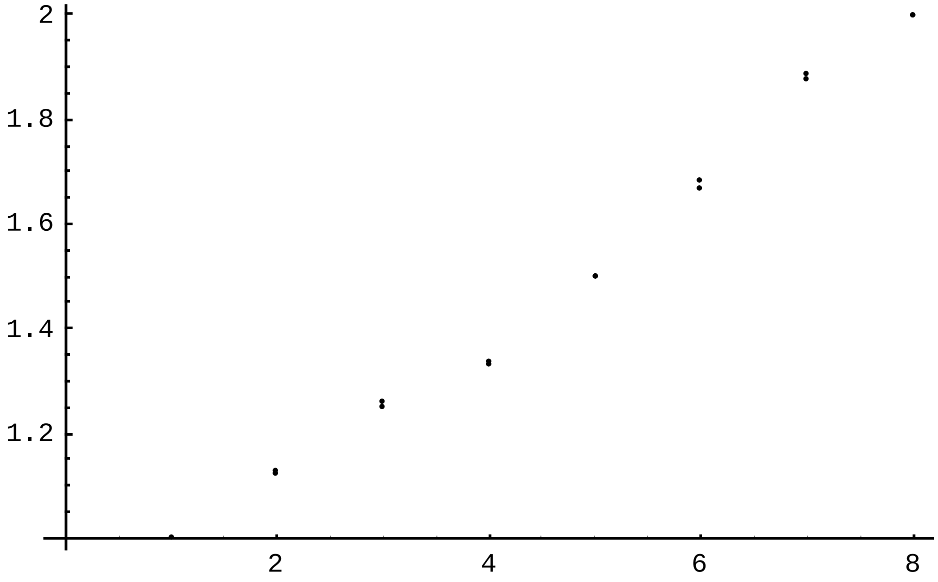
$$R(n) = 2^{\frac{n}{12}}$$

0 - 1.00000,
1 - 1.05946,
2 - 1.12246,
3 - 1.18921,
4 - 1.25992,
5 - 1.33484,
6 - 1.41421,
7 - 1.49831,
8 - 1.58740,
9 - 1.68179,
10 - 1.78180,
11 - 1.88775,
12 - 2.00000

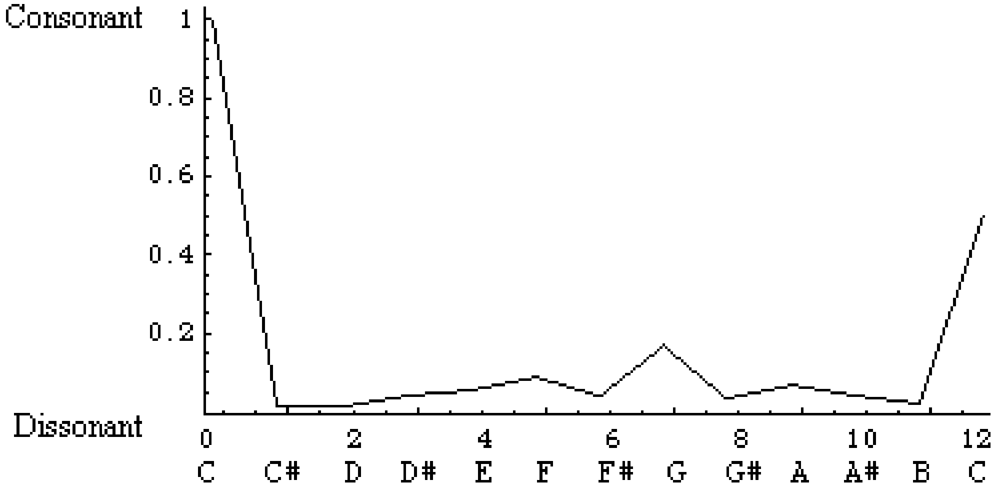


The twelve tone scale is used, rather than one with more or less tones, because its seventh value (1.49831) is very closed 1.5, which is consonant.

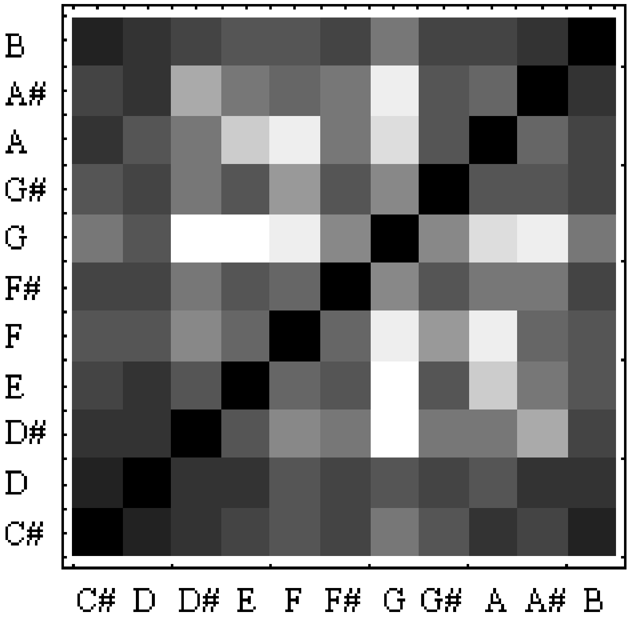
The seven most consonant tones in the equitempered scale make up the major scale, and are said to represent the just intonation scale which consists of pure consonant ratios. However, the equitempered scale does not fit the just intonation scale perfectly, which can be seen when they are plotted together:



It is due to this imperfection that certain chords are not as consonant as might be expected. Unlike other theories of consonance, however, our theory is capable of determining the harmonic value of a chord while taking consideration of imperfections. We have merely to use the ratios of the equitempered scale as input:



The above plot gives the harmonic value of the tone indicated by the x-axis when played with the root.



This plot (h=2000, t=420) shows the harmonic value of a triad made up of two of the 12 equitempered tones and the fundamental (1). For clarity, we have eliminated all chords with a repeated note because these are not triads and are represented in the previous plot (for diatonic chords).

Thus, the most consonant triads in the range shown are the major chord (1:5/4:3/2), the second inversion of the major chord (1:4/3:5/3), a chord consisting of the root, the fourth (4/3), and the fifth (3/2), the minor chord (1:6/5:3/2), the seventh chord without the third (1:3/2:7/4), a chord consisting of the root, the fifth (3/2), and the sixth (5/3), the first inversion of the major chord (1:5/4:5/3), and the minor seventh (1:6/5:7/4). The most dissonant chord consists of the root, the minor second, and the major second (9/8).

We can apply this thoery of sensory harmonics to certain visual forms as well as audible. For example, rectangles and bricks have harmonic values which can be determined for a given ratio of their sides. The following rectangles are shown in decreasing order of their harmonic values:

It is evident, especially in the above visual example, that the harmonic value does not conincide perfectly to aesthetic value. In fact, most people would attest to the third rectangle as being the most aesthetically pleasing. According to Birkhoff, several rectancles have historically been hailed as being the most aesthetically pleasing. The ratios of these rectangles include:

$$\sqrt{3}:1$$

$$\sqrt{2}:1$$

$$\frac{\sqrt{5}-1}{2}:1$$

The third relationship is called the golden section and is commonly used in an aesthetic context. We shall say that this and other rectangles have high aesthetic value (*Av*) due to their harmonic value and their level of interest, or variance. We shall say that this variance or interest is equivalent to dissonance. Thus we have observed a paradox of aesthetics, we have shown that both consonance and dissonance contribute to aesthetic value.

To represent this relationship mathematically, we shall assume that an ideal ratio of consonance to dissonance, or an ideal harmonic value, exists. We can now derive a function which describes the relationship between harmonic value and aesthetic value, using an aesthetic constant (*A*) which indicates the ideal harmonic value. This function will have the same properties as a bell curve and, in fact, has many of the same properties as harmonic nearness. For this reason, we shall estimate the relationship with harmonic nearness where $h=1$. In addition, we must account for the fact that very large and very small ratios have low aesthetic value due to the difficulty in distinguishing a relationship between them. We shall use harmonic nearness again to represent this effect.

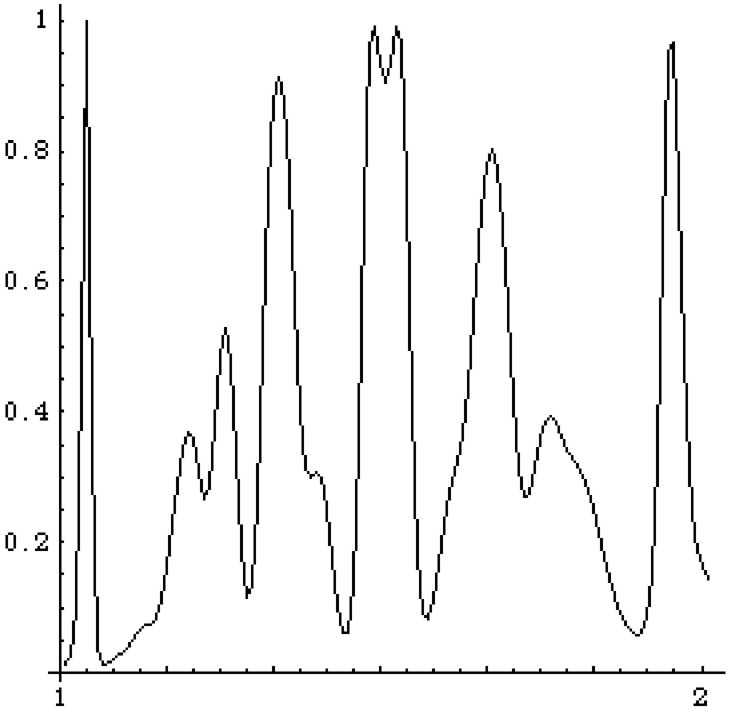
Theorom 8.

$$Av(s) = Hn\left(Hv(s), A\right)\cdot Hn(s,1)$$

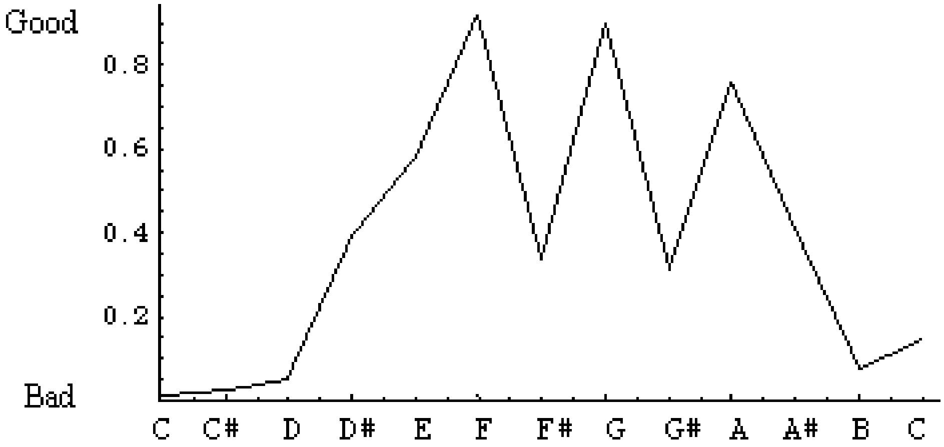
or

$$Av(s) = e^{-\left(\ln^2 \frac{Hv(s)}{A}\right)} e^{-\left(\ln^2 s\right)}$$

$$Av(s) = e^{-\left(\ln^2 \frac{Hv(s)}{A}\right)-\left(\ln^2 s\right)}$$



t=120, A=1/8



t=420, A=1/8

The above plots represent the aesthetic values of a diatonic chord.

Works Cited/Further Reading

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